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# A Closed-Form Solution for the Ground Response Curve of Circular Tunnels Considering Large Deformations

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## Abstract

The analysis of the ground response to tunnel excavation is described usually in terms of the characteristic line of the ground, which relates the support pressure with the cavity wall displacement. Squeezing conditions may lead to large convergences, sometimes greater than 10-20% of the tunnel radius, while the majority of the existing formulations for the Ground Response Curve (GRC) are based on the theory of infinitesimal deformations. This paper presents a large strain analytical solution for the GRC considering a linearly elastic-perfectly plastic material that obeys the Mohr-Coulomb failure criterion with a non-associated flow rule. The case of out-of-plane plastic flow is included, taking place when the longitudinal stress is no longer the intermediate principal one. Comparisons with the classic small strain solution as well as with an approximate large strain solution which neglects elastic deformations in the plastic zone are presented, demonstrating moreover the influence of plastic dilatancy. The derived relations are found in perfect agreement with finite element results; hence, apart from their usefulness in convergence assessments when extreme squeezing conditions are expected, they can provide also valuable benchmark for numerical procedures which take into account large deformations.

**Keywords:** Closed-form solution, dilatancy, ground response curve, large strain, squeezing

## 1 INTRODUCTION

Large convergences are encountered often in underground projects, which combine high overburden with poor ground properties. Various reports can be found in the literature dealing with tunnel cases under heavily squeezing conditions, e.g. the Gotthard base tunnel in Switzerland [3]. A widely used method for the estimation of the ground behaviour during tunnel excavation considers the characteristic line of the ground, which relates the support pressure with the wall radial displacement. A circular tunnel cross-section under axisymmetric plane strain conditions is usually used as the static system, corresponding reasonably well to the situation that prevails in deep tunnels far behind the face.

The majority of the existing analytical solutions concerning the GRC are based on the small deformation theory taking into consideration several elastoplastic constitutive models with different post-failure behaviour. The most common of them for rock-like materials are the elastic-perfectly plastic, the elastic-brittle plastic and the strain softening one. Furthermore, some viscous models have been developed in order to account for time-dependent effects, which may be significant in case of squeezing rocks with considerable rheological behaviour, e.g. creep.

On the contrary, few attempts have been made for the derivation of the GRC accounting for finite deformations. Papanastasiou and Durban [4] studied both problems of expanding and contracting cylindrical cavities in an infinite isotropic medium using the Mohr-Coulomb (MC) as well as the Drucker-Prager hardening solid, resulting in differential equations which must be solved numerically. Their results found to be in good agreement with experimental ones. Later, Yu and Rowe [7] presented an analytical solution to the problem of cavity wall unloading (for cylindrical or spherical cavities) using the linearly elastic-perfectly plastic MC model. However, in order to express the cavity contraction curve in closed form, they ignored elastic deformations within the plastic zone. Vrakas and Anagnostou [6] extended recently this study, presenting an accurate solution to the problem without neglecting elastic deformations in the plastic region, considering the more general elastic-brittle plastic case as well as examining the influence of the out-of-plane stress (for the two-dimensional case). It is shown that when this stress is no longer the intermediate principal one, an inner second plastic ring is formed, where an edge flow takes place, similarly to the small strain case presented by Reed [5].

The presented paper is based upon the extended report [6]. It outlines the derivation of the large strain GRC, summarizes the equations for the GRC, shows the validity limit of the small strain assumption and compares the closed-form solution with numerical results. It shall be noted here, that this study focuses on fully drained cases without seepage flow or dry grounds, where the influence of the water is negligible.

## 2 LARGE STRAIN CLOSED-FORM SOLUTION FOR THE GRC

The classic problem of a cylindrical cavity with radius $a_o$, unloaded from the in situ state of stresses in an infinite medium is examined (Fig. 1). An isotropic initial stress field expressed by the stress $\sigma_o$ is considered, while the gravitational forces are neglected; therefore, the problem becomes one-dimensional with respect to the radial direction. During unloading, the radial, $\sigma_r$, and the tangential stress, $\sigma_t$, constitute the minor and the major principal stress, respectively, due to the convention of positive compression that is used. It shall be noted, that the considered stresses correspond to the Cauchy ones (i.e., force per current unit area), while the appropriate logarithmic definition is adopted for the strains.

As the internal support pressure, $\sigma_a$, is reduced successively, the ground behaviour around the opening is initially purely elastic. The radial stresses decrease while the tangential ones increase until the MC criterion is satisfied at the tunnel wall. This takes place when the critical value $\sigma_{\rho 1}$ is reached. Then, the material starts to become plastified forming a plastic zone of radius $\rho_1$ around the cavity (Fig. 1). The higher order terms of the Hencky strains during elastic response are neglected, otherwise a closed-form analytical solution could not be obtained, except for an incompressible material. This approximation leads to the classic small strain relations for the elastic domain with respect to the Eulerian (or spatial) radial coordinate $r$.

Accounting for the equilibrium equation on the deformed configuration, the MC failure criterion, the boundary condition at the cavity wall and the continuity of stresses at the elastoplastic interface, the stresses inside the plastic ring can be expressed in terms of the ratio $r/a$, where $a$ denotes the current tunnel radius. In contrast to the small strain formulation [5], the determination of the displacement field is a prerequisite here for the estimation of the stress field. Considering moreover the plastic flow rule and the continuity of radial displacements at the elastoplastic boundary, the cavity contraction curve can be calculated after some mathematical treatment. The axial stress is obtained by the corresponding elastic constitutive relation ($\varepsilon_z = 0$).

As the support pressure $\sigma_a$ is further reduced to the value $\sigma_{\rho 2}$ (ensuring that $\sigma_{\rho 2}$ is positive, which is true under certain conditions concerning the problem parameters, see Eq. 3), a second inner plastic ring of radius $\rho_2$ begins to form, where the longitudinal stress, $\sigma_z$, remains equal to the tangential one. Inside this region, an out-of-plane plastic flow takes place. The radial as well as the tangential stresses are still given by the same relationships in both zones as in the previous case, whereas the continuity of displacements at radii $\rho_1$ and $\rho_2$ in combination with the appropriate flow rule in each zone must be considered for the estimation of the displacement field.

The necessary relations for the construction of the GRC accounting for large strains are given below. The general expressions as well as the complete mathematical process for their derivation can be found in [6]. The corresponding small strain solution of the problem is presented in [5]. Hence, regarding that the elastic material properties are expressed by the Young's modulus, $E$, and the Poisson's ratio, $v$, while the plastic ones are given by the cohesion, $c$, the friction angle, $\varphi$, and the dilation angle, $\psi$, the tunnel wall displacement (positive inwards), $u_a$, can be calculated:

$$\frac{u_a}{a_o} = \begin{cases} \left[1 + \dfrac{E/\left(1+v\right)}{\sigma_o - \sigma_a}\right]^{-1} & , \quad \sigma_a \geq \sigma_{\rho 1} \\ 1 - \left[T_{oi} + \left(\delta/\Omega_{oi}\right) \times \mathrm{f}_i\left(1, R_i\right)\right]^{-\frac{1}{\kappa+1}} & , \quad \begin{array}{l} \sigma_{\rho 2} \leq \sigma_a < \sigma_{\rho 1} \;\; \left(i=1\right) \\ \sigma_a < \sigma_{\rho 2} \;\; \left(i=2\right) \end{array} \end{cases} , \tag{1}$$

where $$m = \frac{1+\sin\varphi}{1-\sin\varphi}, \quad \sigma_D = \frac{2c\cos\varphi}{1-\sin\varphi}, \quad \kappa = \frac{1+\sin\psi}{1-\sin\psi}, \quad \delta = \frac{\kappa+1}{m-1}, \tag{2}$$

$$\sigma_{\rho 1} = \frac{2\sigma_o - \sigma_D}{m+1}, \quad \sigma_{\rho 2} = \frac{\left(1-2v\right)\sigma_o - \left(1-v\right)\sigma_D}{m - v\left(m+1\right)}, \tag{3}$$

$$\overline{x} = x + \frac{\sigma_D}{m-1} \text{ (abbr.)}, \quad R_i = \frac{\overline{\sigma}_{\rho i}}{\overline{\sigma}_a}, \tag{4}$$

$$\omega_{11} = \frac{1+v}{E}\left[1 - v\left(\kappa+1\right)\right], \quad \omega_{12} = \frac{1}{E}\left(1 - 2v\kappa\right), \tag{5}$$

$$\omega_{21} = \frac{1+v}{E}\left[\kappa\left(1-v\right) - v\right], \quad \omega_{22} = \frac{2}{E}\left[\kappa\left(1-v\right) - v\right], \tag{6}$$

$$\Omega_{oi} = \exp\left[\left(\omega_{1i} + \omega_{2i}\right)\overline{\sigma}_o\right], \quad \Omega_i = \left(\omega_{1i} + m\omega_{2i}\right)\overline{\sigma}_a, \tag{7}$$

$$T_{o1} = \left[ 1 + \frac{\sigma_o - \sigma_{\rho 1}}{E / (1+v)} \right]^{\kappa+1} R_1^{\delta}, \quad T_{o2} = T_{o1} + \frac{\delta}{\Omega_{o1}} \mathrm{f}_1 (R_2, R_1), \tag{8}$$

$$\mathrm{f}_i (x, y) = \sum_{n=0}^{\infty} \frac{\Omega_i^n}{n! (n+\delta)} \left( x^{n+\delta} - y^{n+\delta} \right) \quad . \tag{9}$$

By neglecting elastic deformations in the plastic zone(s), the solution is simplified to a large degree, given by a single relation during elastoplastic response [7]:

$$\frac{u_a}{a_o} = \begin{cases} \left[ 1 + \dfrac{E / (1+v)}{\sigma_o - \sigma_a} \right]^{-1} & , \quad \sigma_a \geq \sigma_{\rho 1} \\ 1 - \left[ 1 - R_1^{\delta} + T_{o1} \right]^{-\frac{1}{\kappa+1}} & , \quad \sigma_a < \sigma_{\rho 1} \end{cases} \quad . \tag{10}$$

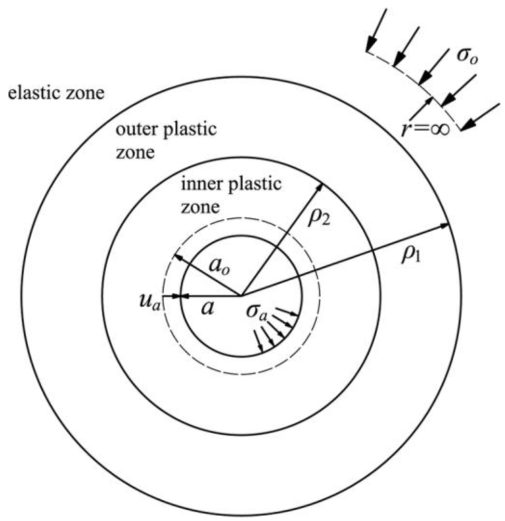


**Figure 1:** Computational model for a deep circular tunnel, with the developed plastic zones.

## 3 ANALYTICAL AND NUMERICAL RESULTS

An application of the aforementioned relations is presented here, in conjunction with some numerical results obtained with the Abaqus software [1]. The considered finite element model is the widely used axisymmetric strip with the proper vertical displacement restraints. The computational domain is discretized by using quadrilaterals elements (CAX4), while one infinite element (CINAX4) is incorporated in order to simulate the unbounded far field (Fig. 2). The classic MC model describes the material behaviour, having a pyramidal failure surface and plastic potential in the principal stress space as well as satisfying the Koiter's rule [2] in case of a singularity stress state, i.e. the incremental plastic strain is given by the sum of the components of the two flow rules.

The computational examples consider the rock properties proposed by Kovári et al. [3] for the Sedrun section of the Gotthard base tunnel in Switzerland, based on experimental investigations of the ETH Zurich: $E$ = 2000 MPa, $c$ = 0.25 MPa and $\varphi$ = 23$^{o}$. The Poisson's ratio $v$ is taken equal to 0.25. The overburden depth is approximately 900 m, which corresponds to an initial stress $\sigma_o$ = 22.5 MPa.

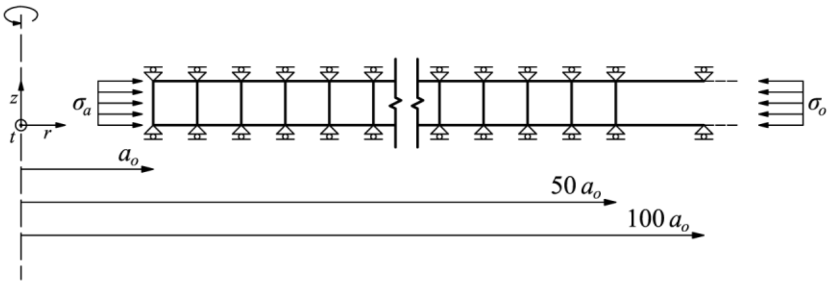


**Figure 2:** Axisymmetric finite element model for the calculation of the GRC.

Figure 3 presents the GRCs for a non-dilatant ($\psi$ = 0$^{o}$) and a dilatant material ($\psi$ = 10$^{o}$ and $\psi$ = 20$^{o}$, respectively). The finite element results fit almost perfectly the analytical ones for both formulations, while the approximate large strain solution that ignores elastic deformations in the plastic zone underestimates the tunnel wall displacements. It can be seen in these graphs that the error of neglecting the elastic deformations in the plastic zone is bigger at the lower dilation angle range. The error of the classic small strain solution is small up to convergence ratios $u_a/a_o$ of 10%, a limit value which in the present example still corresponds to significant support pressures. One recognizes, furthermore, that the small strain solution may lead to irrational results, providing convergences greater than the initial tunnel radius (Fig. 3b-c). Finally, it can be confirmed, that the dilatancy favours the developed displacements to a great extent due to the unrestrained volumetric increase of the assumed constitutive model.

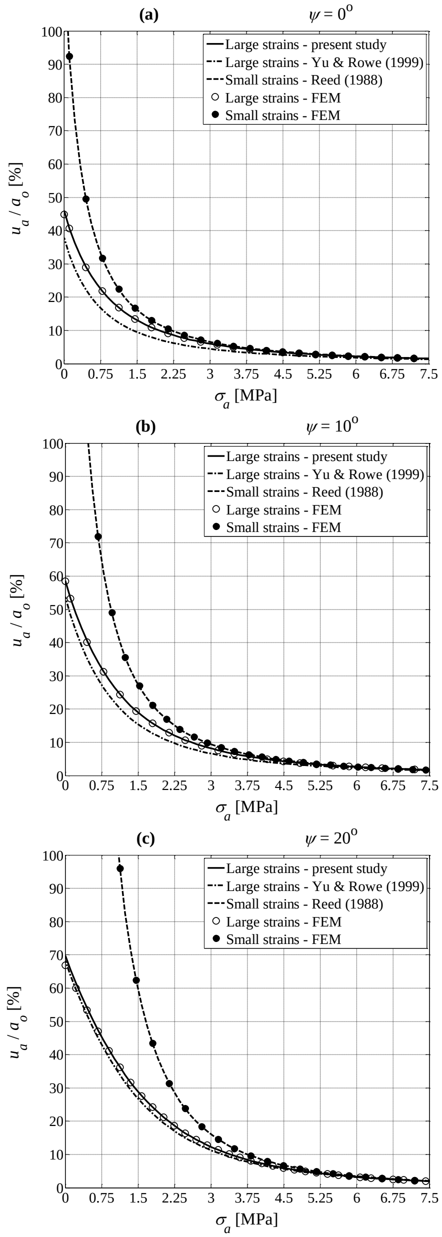


**Figure 3:** Ground response curves for a dilation angle: (a) $\psi = 0^o$, (b) $\psi = 10^o$ and (c) $\psi = 20^o$.

## 4 CONCLUSION

A closed-form solution for the GRC was presented considering finite strains. The proposed analytical expressions can be used for convergence assessments in tunnelling under extreme squeezing conditions, while they constitute also trustworthy benchmark for numerical procedures which account for geometric nonlinearities.

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